LINEAR ALGEBRA HOMEWORK

JULY 31, 2023

Exercise 1. *Given a matrix A, define the row space of A as*

$$Row(A) = \text{span of the rows of } A.$$

Prove that each ERO does not change row space of a matrix. So we have if $B \sim A$, then $Row(B) = Row(A)$.

Exercise 2. *By Proposition in class, there exist $v_1', \ldots, v_s' \in \mathcal{V}$ such that $f(v_1'), \ldots, f(v_s') \in \mathcal{U}$ form a basis of $im f$. Let $v_1, \ldots, v_r \in ker f \subset \mathcal{V}$ be a basis of $ker f$. Prove that these $r+s$ vectors in $\mathcal{V}$ form a basis of $\mathcal{V}$. Then prove the equation of conservation of dimensions holds.*

Exercise 3. *Assume $dim_F\mathcal{V}$ is finite and $\mathcal{W}$ is a subspace of $\mathcal{V}$. Compute $dim_F\mathcal{V}/\mathcal{W}$. (Write in 3 lines.)*

Exercise 4. *Let*

$$\begin{array}{rccc} \varphi : & F^n & \xrightarrow[\sim]{} & \mathcal{V} \\ & e_i & \longmapsto & v_i \end{array}$$

be a linear bijection. Then φ can be identified as $\varphi \equiv (v_1, \ldots, v_n)$, and $v_1, \ldots, v_n$ form a basis of $\mathcal{V}$.

For any $f \in \operatorname{End}\mathcal{V}$, and any $1 \leq i \leq n$, $f(v_i) \in \mathcal{V}$. Then there exists unique $f_{1i}, \ldots, f_{ni} \in F$ such that

$$f(v_i) = \sum_{j=1}^{n} f_{ji} v_j.$$

In this way give any $f \in \operatorname{End}\mathcal{V}$, we can define a unique $(f_{ji}) \in M_n$. Therefore we have a map

$$\begin{array}{rccc} \Phi : & \operatorname{End}\mathcal{V} & \longrightarrow & M_n \\ & f & \longmapsto & (f_{ji}). \end{array}$$

Prove that Φ is an isomorphism, i.e. Φ is a linear bijection such that

(1) $\Phi(1) = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$;
(2) $\Phi(ff') = \Phi(f)\Phi(f')$;
(3) $\Phi(f + f') = \Phi(f) + \Phi(f')$.